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MAKING EXCHANGE RATES SPARKLE: RESTRICTING ITS PRESENT-VALUE MODEL WITH COMMON TRENDS AND COMMON CYCLES

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Abstract

Exchange rates have raised the ire of economists for more than 20 years. A problem is that there appears to be no exchange rate model that systematically beats a naive random walk in out of sample forecasts. Economists also find it irksome that theoretical models are unable to explain short-, medium-, and long-run exchange rate movements. This paper shows that the present value model (PVM) imposes *common trend* and *common cycle* restrictions on the exchange rate and its $I(1)$ fundamental. A theoretical implication is that the exchange rate approximates a martingale, as the interest sensitivity of money demand grows large. Along with this restriction, we also find that a common cycle is necessary for the exchange rate to approximate an independent random walk. A PVM of the exchange rates is also constructed from a dynamic stochastic general equilibrium (DSGE) open economy model. The DSGE-based PVM predicts that the exchange rate and fundamentals are co-dependent because their impulse response functions are collinear after a finite horizon. Thus, our results indicate the PVM can account for exchange rate fluctuations, while helping to understand why the random walk model does so well, which complements Engel and West (2004, 2005) and presents a new challenge to future research.

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Key Words : Exchange Rates; Present-Value Model; Common Trends; Common Cycles.

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1. Introduction

The search for satisfactory exchange rate models continues to be elusive. Since the seminal work of Meese and Rogoff (1983a, 1983b), a variety of models have been tried in an effort to improve on naive random walk forecasts of exchange rates. These range from linear rational expectations models examined by Meese (1986) to nonlinear models proposed by Diebold and Nason (1990), Meese and Rose (1991), Gençay (1999), and Kilian and Taylor (2003).

The Journal of International Economics volume edited by Engel, Rogers, and Rose (2003) indicates that there has been a split between theoretical exchange rate models and what is considered a useful forecasting model. For example, Kilian and Taylor (2003) argue that there are specific nonlinear forecasting models that can vie with a naive random walk of exchange rates. However, their motivation is empirical only, bereft of theory. This approach maybe useful to obtain candidates for a forecast competition. Nonetheless, there are limits because, as Diebold and Nason (1990) note, the class of nonlinear exchange rate models might be infinite.

This paper takes a step back from the exchange rate forecasting problem to study a workhorse theory of exchange rate determination, the present-value model (PVM) of exchange rates. Actual data most often rejects the exchange rate PVM. Typical are tests Meese (1986) reported that are based on the first ten years of the floating rate regime. He finds that exchange rates are infected with persistent deviations from fundamentals, which reject the PVM. However, Meese is unable to uncover the source of the rejections. Rather than a condemnation of the PVM, we view results such as Meese's as a challenge to update and deepen its analysis.

A similar position is taken by Engel and West (2004, 2005). They explain the random walk behavior of exchange rates and the puzzle as to why alternative models have difficulty competing against it. Starting with the PVM and using uncontroversial assumptions about fundamentals and the discount factor, Engel and West (EW) show that the PVM predicts exchange rates approximate a random walk if currency traders are highly interest sensitive and fundamentals are $I(1)$. They also report empirical and simulation evidence consistent with their theoretical results.

This paper complements Engel and West (2004, 2005). The exchange rate is shown to follow a random walk, but for reasons that differ from EW's. The random walk behavior of exchange rates is tied to the common cycle exchange rates and fundamentals share. We argue that neglect of this common cycle is a source of the failure of the PVM of the exchange rate. Thus, this paper takes up the challenge to update and deepen the PVM.

We extend the PVM of the exchange rate with eight propositions. The propositions are based on PVM cross-equation restrictions that can be interpreted as common trends and common cycles. The PVM cross-equation restrictions rely only on the assumptions that fundamentals are $I(1)$ and possess a Wold representation in first differences.

Theoretical and testable propositions are constructed from the workhorse PVM and an optimizing model of exchange rates. The workhorse PVM yields propositions that: (1) there is a cointegration relationship between the exchange rate and fundamentals [Campbell and Shiller (1987)]; (2) the PVM cross-equation restrictions imply an error correction mechanism (ECM) for currency returns in which the lagged cointegrating relation is the only regressor; (3) if fundamental growth depends only the lagged ECM, the exchange rate and fundamental share a common trend and a common cycle in the sense of Vahid and Engle (1993), (4) the PVM predicts a limiting economy (*i.e.*, the interest rate semi-elasticity of money demand becomes infinite) in which the exchange rate is a martingale, and (5) the EW random walk result can be interpreted as the limiting economy of (4) along with the restriction that the bivariate exchange rate-fundamental process fails to cointegrate, but shares a common cycle.

The three remaining propositions apply to an optimizing dynamic stochastic general equilibrium (DSGE) model of exchange rate determination. Once again, we find that the exchange rate cointegrates with fundamentals when the latter are $I(1)$. The DSGE model makes cross-country consumption differentials a part of fundamentals, rather than cross-country output differentials as in the workhorse PVM. Likewise, the exchange rate and fundamentals share a common cycle if the transitory component of cross-country consumption differentials is restricted to be white noise. Otherwise, the exchange rate and fundamentals are co-dependent in the sense of Vahid and Engle (1997). This implies transitory movements in exchange rates and fundamentals are imperfectly synchronized up to a finite horizon, after which their cycles are common. We propose exploring the predictions of the DSGE model for exchange rate determination with the Kalman filter and the associated maximum likelihood estimator, which follows Schleicher (2006). Schleicher also develops an algorithm to compute a Beveridge and Nelson (1981) and Stock and Watson (1988) trend-cycle decomposition. This Beveridge and Nelson, Stock and Watson (BNSW) decomposition provides information about the relative contribution of the common trend and common cycles to exchange rate fluctuations.

The outline of the paper follows. The next section solves the PVM of the exchange rate and presents its common trend and common cycle restrictions. Section 3 sketches our econometric strategy. Results will be presented in section 4. We conclude in section 5.

2. The Present-Value Model of Exchange Rates

Our model of exchange rate determination combines a liquidity-money demand function, uncovered interest rate parity (UIRP), purchasing power parity (PPP), and flexible prices. This is a workhorse exchange rate model used by, among others, Dornbusch (1976), Frankel (1979), Bilson (1978), Frenkel (1979), Meese (1986), Mark (1995) and Engel and West (2004, 2005).

2a. The Model

Our analysis starts with the liquidity-money demand function

(1) $$m_{h,t} - p_{h,t} = \psi y_{h,t} - \phi r_{h,t}, \quad 0 < \psi, \phi,$$

where $m_{h,t}$ $p_{h,t}$, $y_{h,t}$, and $r_{h,t}$ denote the home country's money stock, aggregate price level, output, and the nominal interest rate. The first three variables are transformed by the natural logarithm. The parameter ψ measures the income elasticity of money demand. Since the nominal interest rate is in its level, ϕ is the interest rate semi-elasticity of money demand. Define the cross-country differentials $m_t = m_{h,t} - m_{f,t}$, $p_t = p_{h,t} - p_{f,t}$, $y_t = y_{h,t} - y_{f,t}$, $r_t = r_{h,t} - r_{f,t}$, where f denotes the foreign country. Assuming PPP holds, $e_t = p_t$, where e_t is the log of the (nominal) exchange rate in which the U.S dollar is the home country's currency.

Under UIRP, the law of motion of the exchange rate is approximately

(2) $$\mathbf{E}_t e_{t+1} - e_t = r_t.$$

Substitute for the nominal interest rate differential in the law of motion of the exchange rate (2) with the liquidity demand function (1) to produce the Euler equation

(3) $$\left[1 - \frac{\phi}{1+\phi}\mathbf{E}_t\mathbf{L}^{-1}\right] e_t = \frac{1}{1+\phi}[m_t - \psi y_t], \quad \mathbf{L}e_t = e_{t-1}.$$

Iterate on Euler equation (3) through date T, recognize the transversality condition

$$lim_{T\to\infty}\left[\frac{\phi}{1+\phi}\right]^{T+1}\mathbf{E}_t e_{t+T} = 0$$

and obtain the present-value relation

(4) $$e_t = \frac{1}{1+\phi}\sum_{j=0}^{\infty}\left[\frac{\phi}{1+\phi}\right]^j \mathbf{E}_t z_{t+j},$$

where the (log) of the exchange rate equals the annuity value of the (log) level of the fundamentals, $z_t \equiv m_t - \psi y_t$. In the PVM, the fundamental z_t is the cross-country money stock differential netted for its income demand component. Also, note that the present-value relation (4) yields the weak prediction that the exchange rate Granger-causes the fundamental $m - \psi y$, a prediction that is explored by Engel and West (2005).

2b. Cointegration Restrictions

The present-value relation (4) provides several predictions given

Assumption 1: $z_t \sim I(1)$.

Assumption 2: $(1 - \mathbf{L})z_t$ has a Wold representation, $(1 - \mathbf{L})z_t = \Delta z^* + \zeta(\mathbf{L})\upsilon_t$.[1]

Given Assumptions 1 and 2, the first prediction is that e_t and z_t share a common trend. This follows from subtracting the latter from both sides of the equality of the present-value relation (4) and combining terms to produce the ECM

$$(5) \qquad e_t - z_t = \sum_{j=1}^{\infty} \left[\frac{\phi}{1+\phi} \right]^j \mathbf{E}_t \Delta z_{t+j}, \quad \Delta \equiv (1 - \mathbf{L}).$$

The ECM reflects the forces that push the exchange rate toward long-run PPP.

Proposition 1: *If z_t satisfies Assumptions 1 and 2, $\mathcal{X}_t = \beta' q_t$ forms a cointegrating relation with cointegrating vector $\beta' = [1 \quad -1]$, where $q_t \equiv [e_t \quad z_t]'$.*

The proposition is a variation of results found in Campbell and Shiller (1987). Note that the cointegrating relation becomes $\mathcal{X}_t = \zeta\left(\frac{\phi}{1+\phi}\right)\upsilon_t$, under Assumptions 1 and 2.

The cointegrating relation $\mathcal{X}_t$ equals the expected present discounted value of Δm_t minus $\psi \Delta y_t$. Thus, $\mathcal{X}_t$ is stationary, given Assumption 1 (*i.e.*, m_t and y_t are $I(1)$ and fail to share a common trend). We interpret $\mathcal{X}_t$ as the 'adjusted' exchange rate because it eliminates cross-country money stock movements netted for its income demand. The 'adjusted' exchange

[1] The restrictions on the moving average are Δz^* is linearly deterministic, $\zeta_0 = 1$, $\zeta(\mathbf{L})$ is an infinite order lag polynominal with roots outside the unit circle, the ζ_is are square summable, and υ_t is mean zero, homoskedastic, linearly independent given history, and is serially uncorrelated with itself and the past of Δz_t.

rate is a forward-looking function of the expected path of fundamental growth. This suggests the cointegrating relation is a "*cycle generator*", as described by Engle and Issler (1995), with the serial correlation of fundamental growth its source.

2c. Equilibrium Currency Return Dynamics

The second PVM prediction begins by writing the present-value relation (4) as

$$e_t - \frac{1}{1+\phi} z_t = \frac{1}{1+\phi} \sum_{j=1}^{\infty} \left[\frac{\phi}{1+\phi} \right]^j \mathbf{E}_t z_{t+j}.$$

Next, difference this equation,

$$\Delta e_t - \frac{1}{1+\phi} \Delta z_t = \frac{1}{1+\phi} \sum_{j=1}^{\infty} \left[\frac{\phi}{1+\phi} \right]^j \left[\mathbf{E}_t z_{t+j} - \mathbf{E}_{t-1} z_{t+j-1} \right],$$

add and subtract $\mathbf{E}_t z_{t+j-1}$ inside the brackets, and use the present-value relation (5) to find

(6) $$\Delta e_t - \frac{1}{\phi} \mathcal{X}_{t-1} = \frac{1}{1+\phi} \sum_{j=0}^{\infty} \left[\frac{\phi}{1+\phi} \right]^j \left[\mathbf{E}_t - \mathbf{E}_{t-1} \right] z_{t+j}.$$

Equilibrium currency return persistence is tied to the ECM, which acts to restore long-run PPP.

Proposition 2: *The Present-Value Model predicts that the equilibrium generating equation of currency returns is an ECM(0), assuming Proposition 1 holds.*

The present-value relation (6) suggests the ECM(0) regression

(7) $$\Delta e_t = \vartheta \mathcal{X}_{t-1} + u_t, \quad \vartheta = \frac{1}{\phi},$$

where $u_t = \frac{\vartheta}{1+\vartheta} \upsilon_t$ under assumption 2.[2]

Regression (7) provides a simple method to compute a BNSW (*i.e.*, permanent-transitory) decomposition for $\{e_t, \ z_t\}$, when Δz_t is also an ECM(0). Part of the explanation relies on the cointegrating relation $\mathcal{X}_t = e_t - z_t$. The relationship between currency returns and fundamental growth fills in the rest.

[2]The error u_t is also justified if the econometrician's information set is strictly within that of currency traders.

Proposition 3: *Assume fundamental growth follows the ECM(0) $\Delta z_t = \eta X_{t-1} + \varpi_t$, where ϖ_t is a mean zero, serially uncorrelated, and homoskedastic disturbance. Given Proposition 2, q_t has a common feature, $\mathcal{F}_t = \tilde{\beta}'\Delta q_t$, in the sense of Engle and Kozicki (1993), where $\tilde{\beta}' = [1 \quad -\frac{\vartheta}{\eta}]$. The cointegrating and common feature vectors β and $\tilde{\beta}'$ restrict the Beveridge-Neslon-Stock-Watson permanent-transitory decomposition of q_t, as described by Vahid and Engle (1993).*

Restrictions on currency returns and fundamental growth are apparent given the ECM(0) of Δz_t. Stack the ECM(0) of fundamental growth below the regression (7) and pre-multiply by $\tilde{\beta}'$ to obtain $\mathcal{F}_t$. Engle and Kozicki (1993) call $\tilde{\beta}$ a common feature vector because it restricts a linear combination of currency returns and fundamental growth to be unpredictable based on the relevant history (*i.e.*, u_t and ϖ_t are uncorrelated at all non-zero leads and lags). Thus, $\mathcal{F}_t$ satisfies the requirements for a common feature relation. Hecq, Palm, and Urbain (2006) note that the common feature $\mathcal{F}_t$ restricts the spectra of Δq_t to be flat. It explains Hecq, Palm, and Urbain (2000, 2003, 2005) calling $\mathcal{F}_t (= \tilde{\beta}'\Delta q_t)$ a strong form common feature.

Proposition 3 predicts $q_t = [e_t \quad z_t]'$ contains a common feature, given cointegration. This mimics a result in Vahid and Engle (1993).[3] Their result imposes a common trend and a common cycle on the exchange rate and fundamental, which arise from restrictions the cointegrating and common feature relations impose on bivariate ECM(0). The restrictions also drive the BNSW decomposition of q_t. For example, the trend and cycle components of e_t equal $\frac{-\eta}{\vartheta - \eta}\tilde{\beta}' q_t$ and $\frac{\vartheta}{\vartheta - \eta}\beta' q_t$, respectively. It is worth pointing out, as Hecq, Palm, and Urbain (2006) do, that the cycles common to currency returns and fundamental growth occur in the short-, medium-, and long-run. Thus, there is no long-run predictability in the exchange rate, which is at odds with the empirical evidence of Mark (1995).

[3]Vahid and Engle show a $n-$dimension VAR(1) with d cointegrating vectors has $n - d$ common feature vectors.

2d. A Limiting Model of Exchange Rate Determination

Proposition 3 yields the strong form common feature $\mathcal{F}_t$ for currency returns and fundamental growth. Their linear combination is unpredictable based on the history of Δq_t and $\mathcal{X}_{t-1}$. An implication of $\mathcal{F}_t$ is that currency returns and fundamental growth share common short-run dynamics. Thus, short- and long-run exchange rate dynamics do not differ. However, Proposition 2, regression (7), and the strong form common feature $\mathcal{F}_t$ rely on $\phi < \infty$. Thus, the ECM(0) of Proposition 2 and the strong form common feature it suggests leads to

Proposition 4: *The exchange rate approaches a martingale (in the strict sense) as* $\phi \longrightarrow \infty$*, according to the present-value relation* (6) *and Proposition 2.*

Hansen, Roberds, and Sargent (1991) study econometrically rich linear rational expectations models whose equilibrium anticipates Proposition 4. The limiting economy of Proposition 4 has an equilibrium in which the best forecast of e_{t+1} is e_t based on the relevant information set. Thus, Proposition 3 predicts that the equilibrium exchange rate can be a random walk because it is a martingale.[4]

2e. PVM Exchange Rate Dynamics Redux

Engel and West (2005) show that the PVM of the exchange rate yields an approximate a random walk when the interest (semi-)elasticity of money demand grows large. Their result relies not on Proposition 1 as does Proposition 4. Rather, EW invoke Assumptions 1 and 2, the present-value relation (4), the Weiner-Kolmogorov prediction formula, and the *conjecture* that $e_t = \mathsf{a}z_t$ to find currency returns are unpredictable.

The EW hypothesis is $plim_{\phi \longrightarrow \infty}[\Delta e_t - \mathsf{a}\zeta(\mathbf{1})\upsilon_t] = 0$. Its hypothesis test begins with

[4] Maheswaran and Sims (1993) show that the martingale restriction has little empirical content for tests of asset pricing models when data is sampled at discrete moments in time.

$$e_t \;=\; z_{t-1} \;+\; \sum_{j=0}^{\infty}\left[\frac{\phi}{1+\phi}\right]^j \mathbf{E}_t\Delta z_{t+j},$$

which is developed from the present-value relation (4). EW use this equation to construct

$$\Delta e_t \;-\; \mathbf{E}_{t-1}\Delta e_t \;=\; \zeta\left(\frac{\phi}{1+\phi}\right)\upsilon_t,$$

given Assumptions 1 and 2 and the Weiner-Kolmogorov prediction formula. Note that the last equation implies currency returns equal the annuity value of fundamental growth

$$\Delta e_t \;=\; \frac{1}{1+\phi}\sum_{j=0}^{\infty}\left[\frac{\phi}{1+\phi}\right]^j \mathbf{E}_t\Delta z_{t+j}.$$

The last two equations yield

$$\Delta e_t \;=\; \zeta\left(\frac{\phi}{1+\phi}\right)\upsilon_t \;+\; \frac{1}{1+\phi}\sum_{j=0}^{\infty}\left[\frac{\phi}{1+\phi}\right]^j \mathbf{E}_{t-1}\Delta z_{t+j}.$$

By letting $\phi \longrightarrow \infty$, the random walk hypothesis is verified.[5] Thus, EW do not need the cointegration restriction of Proposition 1 to obtain their random walk result.

EW employ Assumptions 1 and 2 to show the exchange rate approaches a random walk as interest sensitivity becomes large. We obtain their result by exploiting a common feature implication of the PVM for currency returns and fundamental growth. We connect this common feature to the EW result with the assumption that Δq_t is $I(0)$ and has a Wold representation, $\Delta q_t = \lambda(\mathbf{L})\xi_t$. When q_t lacks common trends, the exchange rate and fundamental possess a multivariate BN decomposition, $q_t = \lambda(\mathbf{1})\Xi_t + \Lambda(\mathbf{L})\xi_t$, where $\lambda(\mathbf{1})$ has full rank, $\Lambda(\mathbf{L}) = \sum_{i=0}^{\infty}\Lambda_i$, $\Lambda_i = -\sum_{j=i+1}^{\infty}\lambda_j$, and $\Xi_t = \sum_{j=0}^{\infty}\xi_{t-j}$. Since the multivariate BN decomposition in growth rates is

[5] This analysis matches equations $A.3 - A.11$ and the surrounding discussion of Engel and West (2005).

$$(8) \qquad \Delta q_t = \lambda(\mathbf{1})\xi_t + \Delta\Lambda(\mathbf{L})\xi_t,$$

we have

Proposition 5: *The exchange rate-random walk hypothesis of Engel and West (2005) requires that currency returns and fundamental growth share a common feature, as well as $\phi \longrightarrow \infty$.*

The EW hypothesis eliminates the BN cycle, $\Lambda(\mathbf{L})\xi_t$, from equation (8). All that remains to drive Δq_t is $\lambda(\mathbf{1})\xi_t$. Thus, Proposition 5 predicts the exchange rate and fundamental are random walks because serially correlated common cycles are annihilated.

Propositions 3, 4, and 5 shape the common feature restriction that affirms the EW hypothesis. The common feature vector $\widetilde{\beta}' = [1 \;\; -\frac{\vartheta}{\eta}]$ of Proposition 3 eliminates serial correlation from Δq_t in the multivariate BN growth rates representation (8), $\widetilde{\beta}'\Delta q_t = \widetilde{\beta}'\lambda(\mathbf{1})\xi_t$. If $\widetilde{\beta}' \longrightarrow [1 \;\; 0]$ as in Proposition 4, Proposition 5 predicts the exchange rate becomes a random walk independent of fundamentals. Thus, the EW hypothesis is justified by a common feature restriction on short-, medium-, and long-run movements in the exchange rate and fundamentals.

2f. A Weak Test of the PVM of the Exchange Rate

Proposition 3 predicts the serial correlation of currency returns and fundamental growth is explained only by the lagged cointegrating relation X_{t-1}. Since lagged currency returns and fundamental growth play no role, currency returns and fundamental growth form a VECM(0) that implies a VAR(1) in levels. This suggests a test for the levels VAR lag length provides evidence about Proposition 3 and the PVM of the exchange rate. Note that the null of the lag length of a VAR involves no cross-equation restrictions implying a weak test of the PVM.

We estimate level VARs on Canadian, Japanese, U.K., and U.S. data to test the VECM(0) implication of Proposition 3. Likelihood ratio (LR) statistics are calculated to conduct lag length tests using foreign currency-U.S. dollar exchange rate, cross country money, and cross-country output on a $1976Q1$ – $2004Q4$ sample.[6] The Japanese-U.S. and U.K.-U.S. data yield LR tests significant at the ten percent level for a VAR(12), while the LR tests select a VAR(8) at the seven percent level for the Canadian-U.S. sample.[7] Thus, the Canadian, Japanese, U.K., and U.S. data fail to support a weak implication of the common feature restriction of Proposition 3.

3. A DSGE Based Present-Value Model of the Exchange Rate

Propositions 1 – 5 help to interpret evidence on the near random walk behavior of actual exchange rates. For example, Proposition 3 states that the exchange rate has a random walk trend, if currency returns and fundamental growth form a VECM(0). Actual data is unlikely to support this VECM(0) because (a) it is difficult to find that exchange rates and fundamentals cointegrate, as noted by Engel and West (2005) among other, and (b) high-order serial correlation exists in currency returns and fundamental growth of industrialized economies as indicted by the lag length tests reported in the previous section. Although the PVM can explain a random walk in exchange rates, the PVM is not treated well by the data.

Rejection of the PVM is often given as a reason to discard linear rational expectations models of exchange rates. This paper does not. Rather, we use such rejections to motivate construction of an equilibrium generating process for exchange rates to better understand persistent deviations from fundamentals that the workhorse PVM cannot explain.

[6]The money stocks (outputs) are measured in current (constant) local currency units and per capita terms. The VARs include a constant and linear time trend. The LR statistics employ the Sims (1980) correction and have standard asymptotic distribution according to results in Sims, Stock, and Watson (1990).

[7]The LR tests also find a VAR(2), VAR(5), and VAR(9) at the three percent level or better for the Canadian-U.S., Japanese-U.S. and U.K.-U.S. samples, respectively. These results are consistent with the VECMs estimated by MacDonald and Taylor (1993), among others.

In this section, we develop a PVM model of the exchange rate derived from a standard optimizing two-country monetary DSGE model. Our aim is to construct an equilibrium exchange rate model whose short-run and long-run behavior better reflects dynamics in actual data. However, this is an empirical question we address below.

3a. The DSGE Model

The optimizing monetary DSGE model consists of the preferences of domestic and foreign economies and their resource constraints. For the home ($i = h$) and foreign ($i = f$) countries, the former objects take the form

$$\mathcal{U}\left(C_{i,t},\ \frac{M_{i,t}}{P_{i,t}}\right) = \frac{\left[C_{i,t}^{\nu}\left(\frac{M_{i,t}}{P_{i,t}}\right)^{(1-\nu)}\right]^{(1-\kappa)}}{1-\kappa}, \quad 0 < \nu < 1, \quad 0 < \kappa, \tag{9}$$

where $C_{i,t}$ and $M_{i,t}$ denote the ith country's consumption and the ith country's holdings of its money stock. The resource constraint of the home country is

$$B_{h,t}^{h} + s_t B_{h,t}^{f} + P_{h,t}C_{h,t} + M_{h,t} = (1+r_{h,t-1})B_{h,t-1}^{h} + s_t(1+r_{f,t-1})B_{h,t-1}^{f} + M_{h,t-1} + P_{h,t}Y_{h,t}, \tag{10}$$

where $B_{i,t}^{i}$, $B_{i,t}^{\ell}$, $r_{i,t-1}$, $r_{\ell,t-1}$, $Y_{i,t}$, and s_t represent the ith country's nominal holding of its own bonds at the end of date t, the ith country's nominal holding of the ℓth country's bonds at the end of date t, the return on the ith country's bond, the return on the ℓth country's bond, the output level of the ith country, and the level of the exchange rate. The two-country DSGE model is closed with $B_{h,t}^{h} + B_{h,t}^{f} + B_{f,t}^{h} + B_{f,t}^{f} = 0$. This condition forces the world stock of nominal debt to be in zero net supply, period-by-period, along the equilibrium path.

The PVM of the exchange rate assumes that fundamentals are $I(1)$. We satisfy this requirement with processes for labor-augmenting technical change, $A_{i,t}$, or total factor productivity (TFP), and money stocks that satisfy

Assumption 3: $ln[A_{i,t}]$ and $ln[M_{i,t}] \sim I(1),\ i = h, f.$

Assumption 4: Cross-country TFP and money stock differentials are $I(1)$.

Assumptions 3 and 4 impose stochastic trends on the two-country DSGE model.

3b. DSGE-Based UIRP and Money Demand

The home country maximizes its expected discount lifetime utility,

$$\mathbf{E}_t\left\{\sum_{j=0}^{\infty}(1+\rho)^{-j}\mathcal{U}\left(C_{h,t+j},\ \frac{M_{h,t+j}}{P_{h,t+j}}\right)\right\},\quad 0 < \rho,$$

subject to (10). The first-order necessary conditions of economy i yield optimality conditions that describe UIRP and money demand. The utility-based UIRP condition of country i is

$$\mathbf{E}_t\left\{\frac{\mathcal{U}_{C,h,t+1}}{P_{h,t+1}}\right\}(1 + r_{h,t}) = \mathbf{E}_t\left\{\frac{\mathcal{U}_{C,h,t+1}}{P_{f,t+1}}\right\}\frac{(1 + r_{f,t})}{s_t}, \tag{11}$$

where $\mathcal{U}_{C,h,t}$ is the marginal utility of consumption of the home country at date t. Given the utility specification (9), the exact money demand function of country i is

$$\frac{M_{i,t}}{P_{i,t}} = C_{i,t}\left(\frac{1-\nu}{\nu}\right)\frac{1 + r_{i,t}}{r_{i,t}},\qquad i = h, f. \tag{12}$$

The consumption elasticity of money demand is unity, while the (semi-)interest elasticity of money demand is a nonlinear function of the steady state bond return.

The UIRP condition (11) and money demand equation (12) can be stochastically detrended and then linearized to produce a DSGE model version of the law of motion of the exchange rate. Begin by combining the utility function (9) and the UIRP condition (11) to obtain

$$\mathbf{E}_t\left\{\frac{\mathcal{U}_{h,t+1}}{P_{h,t+1}C_{h,t+1}}\right\}(1 + r_{h,t}) = \mathbf{E}_t\left\{\frac{\mathcal{U}_{h,t+1}}{P_{f,t+1}C_{i,t+1}}\right\}\frac{(1 + r_{f,t})}{s_t},$$

where $\mathcal{U}_{i,t}$ is the utility level of country i at date t. Prior to stochastically detrending the previous expression, define $\widehat{\mathcal{U}}_{i,t} = \mathcal{U}_{i,t}/A_{i,t}$, $\hat{P}_{i,t} = P_{i,t}A_{i,t}/M_{i,t}$, $\hat{C}_{i,t} = C_{i,t}/A_{i,t}$, $\gamma_{A,i,t} = A_{i,t}/A_{i,t-1}$, $\gamma_{M,i,t} = M_{i,t}/M_{i,t-1}$, $\hat{s}_t = s_t A_t/M_t$, $A_t = A_{h,t}/A_{f,t}$, and $M_t = M_{h,t}/M_{f,t}$. Note that $\hat{C}_{i,t}$ is the transitory component of consumption of the ith economy, $\gamma_{A,i,t}(\gamma_{M,i,t})$ is the TFP (money) growth rate of country i, and the cross-country TFP (money stock) differential A_t (M_t) are $I(1)$. Applying the definitions, the stochastically detrended UIRP condition becomes

$$\mathbf{E}_t\left\{\frac{\widehat{\mathcal{U}}_{h,t+1}\gamma_{A,h,t+1}^{1-\kappa}}{\gamma_{M,h,t+1}\hat{P}_{h,t+1}\hat{C}_{h,t+1}}\right\}(1+r_{h,t}) \quad = \quad \mathbf{E}_t\left\{\frac{\widehat{\mathcal{U}}_{h,t+1}\gamma_{A,f,t+1}}{\gamma_{A,h,t+1}^{\kappa}\gamma_{M,f,t+1}\hat{P}_{f,t+1}\hat{C}_{h,t+1}}\right\}\frac{(1+r_{f,t})}{\hat{s}_t},$$

where $i = h, f$. A log linear approximation of the stochastically detrended UIRP condition yields

$$\mathbf{E}_t\tilde{e}_{t+1} - \tilde{e}_t \quad = \quad \frac{r^*}{1+r^*}\tilde{r}_t + \mathbf{E}_t\left\{\tilde{\gamma}_{A,t+1} - \tilde{\gamma}_{M,t+1}\right\}, \tag{13}$$

where $\tilde{e}_t = \ln[\hat{s}_t] - \ln[s^*]$ and $r^*(= r_h^* = r_f^*)$ denotes the steady state (or population) world real rate, for example.

The DSGE model produces a log linear approximate law of motion of the exchange rate (13) which includes an unobserved time-varying risk premium, the expected money and TFP growth differentials. Thus, according to our DSGE model, deviations from unobserved fundamentals are attributed to changes in money growth and fluctuations in multi-factor productivity disparities across the domestic and foreign economies.

3c. A DSGE-Based PVM of the Exchange Rate

We use the linear approximate law of motion of the exchange rate (13), and a stochastically detrended version of the money demand equation (12) to produce the PVM of the exchange of our DSGE model. The unit consumption elasticity-money demand equation (12) implies

$$-\tilde{p}_t \quad = \quad \tilde{c}_t - \frac{1}{1+r^*}\tilde{r}_t. \tag{14}$$

Impose PPP on the stochastically detrended version of the money demand equation (14) and combine it with the law of motion (13) of the transitory component of the exchange rate to find

$$\left[1 - \frac{1}{1+r^*}\mathbf{E}_t\mathbf{L}^{-1}\right]\tilde{e}_t = \frac{1}{1+r^*}\mathbf{E}_t\left\{\tilde{\gamma}_{M,t+1} - \tilde{\gamma}_{A,t+1}\right\} - \frac{r^*}{1+r^*}\tilde{c}_t,$$

with its PVM

(15) $$\tilde{e}_t = \sum_{j=1}^{\infty}\left(\frac{1}{1+r^*}\right)^j \mathbf{E}_t\left\{\tilde{\gamma}_{M,t+j} - \tilde{\gamma}_{A,t+j}\right\} - \frac{r^*}{1+r^*}\sum_{j=0}^{\infty}\left(\frac{1}{1+r^*}\right)^j \mathbf{E}_t\tilde{c}_{t+j},$$

where the tranversality conditions are implied by long-run behavior of $\tilde{\gamma}_{M,t}$, $\tilde{\gamma}_{A,t}$, and $\tilde{c}_t$. The PVM relation (15) is the equilibrium law of motion of transitory component of the exchange rate. It equates exchange rate fluctuations to the future discounted expected path of cross-country money and TFP growth and the (negative of the) annuity-value of the transitory component of cross-country consumption. These factors suggest possible sources of serial correlation to explain exchange rate fluctuations.

3d. DSGE Cointegration Restrictions

The DSGE model produces an ECM of the exchange rate. The cointegrating relation follows from the balanced growth restrictions of the DSGE model, $e_t \equiv \ln[s_t] = \ln[\hat{s}_t] + m_t - \ln[A_t]$, where $m_t = \ln[M_t]$. Thus, the DSGE model yields the cointegrating relation

(16) $$x_{DSGE,t} = \tilde{e}_t + \tilde{c}_t, \quad x_{DSGE,t} \equiv e_t - (m_t - c_t),$$

where constants are ignored $c_t = \ln[C_t]$, and stochastic detrending implies $\ln[A_t] = c_t - \tilde{c}_t$.

The ECM reflects the forces that push the exchange rate toward long-run PPP plus sources of short- and medium-run PPP deviations. The persistence of PPP deviations rely on the forward-looking component $\tilde{e}_t$ and transitory date t cross-country consumption, $\tilde{c}_t$. Nonetheless, the DSGE model restricts PPP deviations to be stationary, which suggests

Proposition 6: *If m_t and A_t satisfy Assumptions 3 and 4, $\mathcal{X}_{DSGE,t} = \beta'_{DSGE} q_{DSGE,t}$ forms a cointegrating relation with cointegrating vector $\beta'_c = [1 \quad -1 \quad 1]$, where $q_{DSGE,t} \equiv [e_t \quad m_t \quad c_t]'$.*

The DSGE model predicts a forward-looking cointegration relation, but with a new source of dynamics. Besides the present-value of fundamental growth, date t transitory cross-country consumption and its annuity value creates persistence and volatility in the "*cycle generator*" $\mathcal{X}_{DSGE,t}$ of (16). Thus, the DSGE model introduces a new source of serial correlated short- and medium-run PPP deviations, which are found in the standard PVM.

3e. DSGE Equilibrium Currency Return Dynamics

The DSGE model produces an equilibrium currency return generating equation that departs from the standard PVM (6). The same process that produced the PVM equilibrium currency return generating equation (6), along with a bit of extra algebra, takes us from the PVM of the exchange rate of the DSGE model (15) to the equilibrium currency return generating equation

$$(17) \qquad \Delta e_t - (\Delta m_t - \Delta c_t - \mathcal{X}_{DSGE,t-1}) = \sum_{j=1}^{\infty} \left(\frac{1}{1+r^*}\right)^j \left[\mathbf{E}_t - \mathbf{E}_{t-1}\right] \left\{\gamma_{M,t+j} - \gamma_{A,t+j}\right\} - \frac{r^*}{1+r^*} \sum_{j=0}^{\infty} \left(\frac{1}{1+r^*}\right)^j \left[\mathbf{E}_t - \mathbf{E}_{t-1}\right] \tilde{c}_{t+j} + \tilde{e}_t + \tilde{c}_t,$$

of the DSGE model.

Proposition 7: *The equilibrium currency return generating equation* (17) *predicts* Δe_t, Δm_t, Δc_t, and $\mathcal{X}_{DSGE,t-1}$ *share a weak form common feature, $\mathcal{F}_{DSGE,t} = \tilde{\beta}'_{DSGE}[\Delta q'_{DSGE,t} \;\; \mathcal{X}_{DSGE,t-1}]'$, where $\tilde{\beta}'_{DSGE} = [1 \quad -1 \quad 1 \quad 1]$, only if $\tilde{e}_t$ and $\tilde{c}_t$ are serially uncorrelated.*

Proposition 7 places restrictions on Δe_t, Δm_t, Δc_t and $\mathcal{X}_{DSGE,t-1}$ in the spirit of the weak form common feature of Hecq, Palm, and Urbain (2006). A weak form common feature is unpredictable, conditional on the relevant history, as is the strong form common feature of Engle and Kozicki (1993) and Vahid and Engle (1993). The point of departure between the strong and weak form common features is that the latter contains the lagged ECM. Remember that Proposition 7 requires the transitory component of cross-country consumption to be white noise for the common feature relation $\mathcal{F}_{DSGE,t}$ to be unpredictable.

The long-run properties of the exchange rate (*i.e.*, PPP) are decoupled from its short-run dynamics, according to Proposition 7. Fluctuations in short-run currency returns and fundamentals growth are tied to movements in the lagged ECM, $\mathcal{X}_{DSGE,t-1}$. The common cycle of currency returns, money growth, and consumption growth share the serial correlation of $\mathcal{X}_{DSGE,t-1}$ because it is not annihilated by the weak form common feature vector $\widetilde{\beta}_{DSGE}$. For the same reason, the exchange rate is predictable in the long-run by the levels of cross-country money and consumption which is consistent with Mark (1995). Nonetheless, no transitory serial correlation can exist in fundamentals for the restrictions of Proposition 7 to hold.

The previous section reports tests for the lag length of levels VARs of exchange rates and fundamentals. The tests select VARs of order greater than one because the transitory component of fundamentals drive higher-order serial correlation in exchange rates. The equilibrium generating process of currency returns suggests the source of the serial correlation.

Proposition 8: *Assume $\widetilde{e}_t$ and $\widetilde{c}_t \sim ARMA(k_1, k_2)$ with maximum lag length k_{DSGE}. The linear combination $\mathcal{F}_{DSGE,t}$ is unpredictable beyond lag k_{DSGE}. It follows that the impulse response function of $\Delta q_{DSGE,t}$ is linearly independent for horizons greater than k_{DSGE}.*

Vahid and Engle (1997) and Schleicher (2006) develop the idea of a common feature that creates imperfectly synchronized or co-dependent cycles in VARMAs, VARs, and VECMs. Perfectly synchronized cycles imply impulse response functions that are white noise subsequent to impact and are associated with strong and weak form common features. The impulse response functions of imperfectly synchronized time series are collinear only after a finite forecast horizon.

Proposition 8 predicts currency returns share a co-dependent cycle with cross-country money growth, cross-country consumption growth, and the lagged ECM $\mathcal{X}_{c,t-1}$. The sources of the imperfectly synchronized cycle are transitory fluctuations in the exchange rate and cross-country consumption. Thus, the DSGE model produces a PVM of exchange rates with cross-equation restrictions conditional on a joint DGP for cross-country money and consumption.

Propositions 6, 7, and 8 rest on two implicit assumptions. One is that the world real interest rate is non-zero, $r^* > 0$. The other is that the transitory components of cross-country money, $\widetilde{m}_t$, and consumption, $\widetilde{c}_t$, have Wold representations. Given the balanced growth restriction, A3 and A4 suggest the trends of M_t and A_t are unit roots (with drift), we have $\mu_{t+1} = \mu^* + \mu_t + \varepsilon_{\mu,t+1}$, $\varepsilon_{M,t+1} \sim \mathbf{N}(0, \sigma^2_{\varepsilon_M})$, and $\ln[A_{t+1}] = a^* + \ln[A_t] + \varepsilon_{A,t+1}$ and $\varepsilon_{A,t+1} \sim \mathbf{N}(0, \sigma^2_{\varepsilon_A})$. Since stochastic detrending of cross-country money and cross-country consumption gives $m_t = \mu_{M,t} + \widetilde{m}_t$ and $c_t = \ln[A_t] + \widetilde{c}_t$ (ignoring constants), the PVM (15) becomes

$$(18) \qquad \widetilde{e}_t = \sum_{j=1}^{\infty} \left(\frac{1}{1+r^*}\right)^j \mathbf{E}_t \left\{\widetilde{m}_{t+j} - \widetilde{m}_{t+j-1}\right\} - \frac{r^*}{1+r^*} \sum_{j=0}^{\infty} \left(\frac{1}{1+r^*}\right)^j \mathbf{E}_t \widetilde{c}_{t+j}.$$

If the Wold assumption is maintained and $r^* \longrightarrow 0$, we have

Proposition 9: *Assume $\widetilde{m}_t \sim MA(\infty)$. As the PVM discount factor $\frac{1}{1+r^*} \longrightarrow 1$, $\widetilde{e}_t$ equals the negative of $\widetilde{m}_t$. Thus, the observed exchange rate is dominated by permanent shocks.*

As $r^* \longrightarrow 0$, (18) becomes $\widetilde{e}_t = -\alpha_m(\mathbf{L})\varepsilon_{m,t}$ after applying the Wiener-Kolmogorov prediction formulas where $\widetilde{m}_t \sim \text{MA}(\infty)$ is $\widetilde{m}_t = \alpha_m(\mathbf{L})\varepsilon_{m,t}$.[8] Thus, the exchange rate is

$$e_t = \mu_t + \widetilde{m}_t - \ln[A_t] + \widetilde{e}_t = \mu_t - \ln[A_t].$$

subsequent to decomposing cross-country money and consumption into permanent and transitory components. The DSGE model predicts e_t is driven only by permanent factors as $r^* \longrightarrow 0$,

[8] Sargent (1987) provides the formulas in chapters XI.24 and XII.3.

given $\widetilde{m}_t$ has a Wold representation. If e_t, m_t, and A_t violate the balanced growth restriction, the exchange rate is an independent random walk. Thus, Proposition 9 approximates the Engel and West (2005) hypothesis that the exchange rate mimics a random walk when the discount factor is near one and fundamentals have a unit root.

This section develops a DSGE model-based PVM of the exchange rate. The DGSE model creates short- and medium-run PPP deviations in equilibrium exchange rates with persistence in the transitory components of cross-country money growth and consumption. Although this suggests testable predictions for the DSGE-PVM of the exchange rate, as the world real rate becomes small the economic and statistical content of these predictions shrink. The next section examines whether these predictions matter for the data.

4. Econometric Methods

This section describes the empirical methods employed to estimate the DSGE-PVM model of the exchange rate. First, we develop a multivariate UC-model to connect the permanent and transitory components of the exchange rate and cross-country money, consumption to the related observed prices and aggregates. Next, the UC model is cast in state space form to evaluate the likelihood function of the data. This section also discusses the priors of the parameters of the UC models and outlines the procedure to draw from the posterior distribution.

4a. The UC Model and Its State Space

The linear approximate UIRP (15) places restrictions on the transitory exchange rate process. The cross-equation restrictions vary with the process that drive the transitory components of cross-country money, $\widetilde{m}_t$, and cross-country consumption, $\widetilde{c}_t$. We assume the $\widetilde{m}_t$ is a MA(n), $\widetilde{m}_t = \sum_{j=1}^{n} \alpha_j \mathbf{L}^j \widetilde{m}_{t-j} + \varepsilon_{m,t-j}$, where $\varepsilon_{m,t} \sim \mathbf{N}(0, \sigma^2_{\varepsilon_m})$. For $\widetilde{c}_t$, we specify a AR(k), $\widetilde{c}_t = \sum_{j=1}^{q} \theta_j \mathbf{L}^j \widetilde{c}_{t-j} + \varepsilon_{c,t}$, $\varepsilon_{c,t} \sim \mathbf{N}(0, \sigma^2_{\varepsilon_c})$.

The balanced growth restriction ties long-run exchange rate behavior to the permanent components of cross-country money and consumption. Given observed cross-country money and consumption are the sum of their permanent and transitory components, the UC model has a state space form. We combine the balanced growth restriction with the permanent and transitory decompositions of cross-country money, m_t, and cross-country consumption, c_t, the equilibrium currency return generating equation (15), and the MA(n) of $\widetilde{m}_t$ and AR(k) of $\tilde{c}_t$ to obtain a restricted UC-model of the exchange rate driven by permanent shocks to cross-country money and TFP and transitory fluctuations in cross-country money and consumption.

The state space form of the UC model with transitory cycles in cross-country money and consumption consists of the observation equation

$$(19)\qquad \begin{bmatrix} e_t \\ m_t \\ c_t \end{bmatrix} = \begin{bmatrix} 1 & -1 & \delta_{m,0} & \delta_{m,1} & \ldots & \delta_{m,n} & \delta_{c,0} & \ldots & \delta_{c,k-1} \\ 1 & 0 & 1 & \alpha_1 & \ldots & \alpha_n & 0 & \ldots & 0 \\ 0 & 1 & 0 & 0 & \ldots & 0 & 1 & 0 & \ldots \end{bmatrix} S_t,$$

where $S_t = [\mu_t \;\; \ln[A_t] \;\; \varepsilon_{m,t} \;\; \varepsilon_{m,t-1} \;\; \ldots \;\; \varepsilon_{m,t-n} \;\; \tilde{c}_t \;\; \tilde{c}_{t-1} \;\; \ldots \;\; \tilde{c}_{t-k}]'$, the factor loadings on $\widetilde{m}_t$ and its lags are

$$(20)\qquad \delta_{m,i} = -\frac{1}{1+r^*}\left[\alpha_i - \frac{r^*}{1+r^*}\sum_{j=i+1}^{n}\left(\frac{1}{1+r^*}\right)^j \alpha_j\right], \quad i = 0, \ldots, n, \quad \alpha_0 \equiv 1,$$

and the factor loadings on $\tilde{c}_t$ and $\tilde{c}_t$ are elements of the row vector

$$(21)\qquad \delta_{c,i} = s_c \frac{r^*}{1+r^*}\left[\mathbf{I}_k - \frac{1}{1+r^*}\Theta\right]^{-1},$$

with $s_c = [1 \;\; \mathbf{0}_{1\times k-1}]$ and Θ is the companion matrix of the AR(k) of $\tilde{c}_t$. The system of first-order state equations is

$$
(22) \qquad \mathcal{S}_{t+1} = \begin{bmatrix} \mu^* \\ a^* \\ 0 \\ \vdots \\ 0 \\ \vdots \end{bmatrix} + \begin{bmatrix} 1 & 0 & \dots & 0 & 0 & 0 & \dots & 0 \\ 0 & 1 & \dots & 0 & 0 & 0 & \dots & 0 \\ 0 & 0 & \dots & 0 & 0 & 0 & \dots & 0 \\ \vdots & \vdots & \mathbf{I}_n & \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & \dots & 0 & 0 & \theta_1 & \dots & \theta_k \\ \vdots & \vdots & & \vdots & \vdots & & \mathbf{I}_k & \vdots \end{bmatrix} \mathcal{S}_t + \mathcal{V}_{t+1},
$$

where $\mathcal{V}_{t+1} = [\varepsilon_{\mu,t+1} \;\; \varepsilon_{A,t+1} \;\; \varepsilon_{m,t+1} \;\; 0 \;\; \dots \;\; 0 \;\; \varepsilon_{c,t+1} \;\; 0 \;\; \dots \;\; 0]'$ and $\mathbf{E}\{\mathcal{V}_{t+1}\mathcal{V}'_{t+1}\} = \mathcal{R}$.

4b. The UC Model and Its Likelihood Function

Equations (19) and (22) define the state space model. Harvey (1989) and Hamilton (1994) map state space models into the Kalman filter to evaluate the likelihood of models.[9] Denote the likelihood $\mathcal{L}(\mathcal{Y}_t | \, \Gamma, \; UC(i))$ where $\mathcal{Y}_t = [e_t \;\; m_t \;\; c_t]'$,

$$
\Gamma = [\beta \;\; \alpha_1 \;\; \dots \;\; \alpha_1 \;\; \theta_n \;\; \dots \;\; \theta_k \;\; \mu^* \;\; a^* \;\; \sigma_\mu \;\; \sigma_A \;\; \sigma_m \;\; \sigma_c \;\; \rho_{A,c}]'.
$$

the DSGE-PVM discount factor is $\beta = \dfrac{1}{1+r^*}$, σ_j is the standard deviation of shock innovation to $j = \mu, A, \widetilde{m}$, and $\widetilde{c}$, $\rho_{A,c}$ is the correlation coefficient of innovations to cross-country TFP trend and transitory component of cross-country consumption, and UC_i denotes UC model i with $\widetilde{m}$ and $\widetilde{c}$ cycles, only the $\widetilde{m}$ cycle, or only the $\widetilde{c}$ cycle.

4c. The Data

The sample consists of data from 1976Q1 to 2004Q4, $T = 116$. We have observations on the Canadian dollar – U.S. dollar exchange rate (average of period) from the IFS data bank. The

[9]Harvey, Trimbur, and van Dijk (2005) use Bayesian methods to estimate trends and cycles of aggregate time series, but their analysis is not based on rational expectations models.

Canadian monetary aggregate is equated with M1 in current Canadian dollar, while for the U.S. we use the Board of Governors Monetary Base (adjusted for changes in reserve requirements) in current U.S. dollars. Consumption is the sum of non-durable and services expenditures in constant local currency units for both economies.[10] The aggregate data is seasonally adjusted and converted to per capita units. The data is logged and multiplied by 400, but neither demeaned nor detrended.

4d. Priors

The second column of table 1 lists the priors of Γ. The parameter vector is appended with three parameters, μ_e, τ_e, and δ_A. The first two parameters account for the level and determinist growth rate of the exchange rate, e_t. The priors of μ_e and τ_e are set to capture the deterministic features of the exchange rate. The parameter δ_A is the factor loading on cross-country TFP, $\ln[A_t]$. The balanced growth restriction predicts $\delta_A = -1$. However, there is little information about δ_A. Thus, we select a prior uniform distribution that contains -1.0, as well as values as small as negative ten. If δ_A is small it indicates the inadequacy of the theoretical balanced growth restriction and the impact of permanent fluctuations in cross-country TFP on the exchange rate. Note that the factor loading on the permanent component of cross-country money m_t is normalized to one.

We choose priors of the MA(n) and AR(k) process of $\widetilde{m}_t$ and $\widetilde{c}_t$ based on $n = k = 2$. These lag lengths admit transitory cycles in cross-country money and consumption to have power at the business cycle frequencies, if the data wants. The means of θ_1, θ_2, α_1, and α_2 are selected to guarantee that $\widetilde{m}_t$ and $\widetilde{c}_t$ are stationary. These four parameters are endowed with normal prior distributions with second moments that allow for different types of transitory

[10]This includes Canadian semi-durable expenditures.

behavior in $\widetilde{m}_t$ and $\widetilde{c}_t$. When a draw generates an eigenvalue greater than one for either the MA or AR parameters, the draw is discarded. Priors on the standard deviations of the shock innovations reflect the lack of good information about these shocks. However, we attach a normally distributed prior to the correlation of innovations to $\ln[A_t]$ and $\widetilde{c}_t$. Its mean is negative to capture our prior that $\ln[A_t]$ than c_t. Since we have no information about the extent of the smoothness, the mean is -0.5 with a standard deviation of 0.2 that allows for values close to negative one or zero. Draws less than negative one are ignored. The correlation of innovations to μ_t and $\widetilde{m}_t$ is fixed at zero because our belief that the sources and causes of permanent and transitory monetary shocks are unrelated.

The UC model has only one 'economic' parameter, the discount factor $\beta = \frac{1}{1+r^*}$. We adopt the Engel and West (2005) prior for β. They conjecture that for $\beta \in [0.9,\ 0.999]$ to generate an exchange rate process observationally equivalent to a random walk depends crucially on the data. Hence, our prior on β is constructed to provide information about this conjecture about the time series behavior of the exchange rate. This is reflected by centering the mean of the prior of the normal distribution at 0.95 with a standard deviation 0.025. We toss out draws of β not in $[0.9,\ 0.999]$.

4e. Estimation Methods

The likelihood function of the UC models do not have analytic solutions. We approximate the likelihood $\mathcal{L}(\mathcal{Y}_t | \Gamma,\ UC(i))$ with numerical methods based on the Metropolis-Hastings simulator. Our approach follows Rabanal and Rubio-Ramírez (2005). They exploit Bayesian estimation tools Geweke (1999) develops. The idea is to evaluate $\mathcal{L}(\mathcal{Y}_t | \Gamma,\ UC(i))$ from the random walk Metropolis-Hastings simulator. The result is the posterior distribution of Γ, which is proportion to the likelihood multiplied by the prior. For this draft, we draw $\mathcal{J} = 200,000$

replications from the posterior of a UC-Model.

5. Results

This section reports on the results of our empirical strategy. This draft presents parameter estimates of the UC model with independent transitory components in cross-country money and consumption. In the future, we plan on estimating models UC models with only a common cycle tied to either a MA(2) or AR(2) process. Given posterior distributions, Rabanal and Rubio-Ramírez show how to use the posterior distribution to construct the marginal likelihood to conduct inference across competing models based on a proposal of Geweke (1999). Thus, their tools give a way to generate information about the way the data judges different restrictions on exchange rate dynamics.

5a. Parameter Estimates

Table 1 contains the posterior means of Γ, along with standard deviations of the posterior in parentheses. The key economic parameter is the discount factor β. Its posterior mean of 0.96 is economically sensible. However, a standard deviation of 0.02 suggests a lack of precision in the data about β, as filtered through the UC-model. It is not unreasonable to believe that β is as large as 0.99 or as small as 0.92, according to its 95 percent coverage interval. Thus, the posterior of β suggest the data will find it difficult to distinguish between the UC model and an independent random walk as the source of exchange rate dynamics. This provides support for the Engel and West (2005) conjecture.

The estimates indicate that the MA(2) process of $\widetilde{m}_t$ and AR(2) process of $\widetilde{c}_t$ generate persistence. The posterior means of $\theta_1 = 0.96$, and $\theta_2 = 0.04$ yield a leading eigenvalue of 0.95 from the associated companion matrix. An eigenvalue of 0.91 is produced by the posterior means of $\alpha_1 = 0.54$ and $\alpha_2 = 0.33$. However, the smaller root is -0.36, which points to substantial short-run reversion in $\widetilde{m}_t$ to an own shock. Shock innovations to $\widetilde{m}_t$ are more volatile than to $\widetilde{c}_t$, according to the estimates of $\sigma_m = 1.67$ and $\sigma_c = 0.70$.

The random walk trends of cross-country money and TFP reveal the former to be more persistent than the latter by a factor of five. Cross-country TFP is a relatively smooth process, $\sigma_A = 0.30$, which suggests permanent income dynamics are at work. Since $\rho_{A,c} = -0.60$, it reinforces the view of a smooth $\ln[A_t$ process. Canadian TFP growth lags behind U.S. TFP growth by 0.7 percent per year, on average, because $a^* = 0.18$. The U.S. money stock grows more slowly in Canadian, but $\sigma_\mu = 1.53$ makes the permanent component of cross-country money volatile.

The deterministic components of the exchange rate show the Canadian dollar was far from par and on average depreciated from $1976Q1$ to $2004Q4$. Estimates of μ_e and τ_e are 125.28 and 1.6464, respectively. These estimates set the level of the Canadian dollar-U.S. dollar exchange rate at 1.37, while the Canadian dollar depreciated at an annual rate of 1.6 percent.

The posterior distribution provides a large (in absolute value) factor loading, δ_A, on cross-country TFP. Although μ_t is more volatile than $\ln[A_t]$, the response of the exchange rate to fluctuations in $\ln[A_t]$ are large and far away from the balanced growth restriction. The estimate of δ_A also shows 'excess' sensitivity in the Canadian dollar-U.S. dollar exchange rate, which suggests the importance of real factors in driving its low frequency movements.

Table 2 presents posterior means of the factor loadings on the shocks to $\widetilde{m}_t$, $\varepsilon_{m,t}$ and its lags, and on $\widetilde{c}_t$ and $\widetilde{c}_{t-1}$. The estimated factor loadings reveal that the Canadian dollar-U.S. dollar exchange rate responds more to movements in $\varepsilon_{m,t}$ and its lags than to fluctuations in the transitory component of $\varepsilon_{c,t}$. The implication is that transitory monetary shocks should matter for the exchange rate than real side shocks. However, this depends on the persistence and volatile of $\widetilde{m}_t$ and $\widetilde{c}_t$.

5b. Permanent-Transitory Decompositions

The permanent-transitory decomposition of cross-country money is found in figure 1. Actual cross-country money is plotted as the solid (blue) in the top window of figure 1. Its trend is the (red) dot-dot line computed as the posterior mean by the passing the 200,000 draws of the vector of Γ and the data through the Kalman smoother.[11] The posterior mean of the cross-country money trend is smoother than its observed counterpart. The standard deviation of the growth rate of μ_t is 1.13 compared to 2.37 for m_t.

The bottom window of figure 1 presents the posterior mean of $\widetilde{m}_t$. Rather than generating a cycle in $\widetilde{m}_t$, its posterior mean exhibits sharp short-run reversion in response to an own shock. For example, the first element of the autocorrelation function (ACF) of the posterior mean of $\widetilde{m}_t$ is -0.09. Note also that the volatility of μ_t is less than $\widetilde{m}_t$'s.

The UC model generates a permanent-transitory decomposition of cross-country consumption with an economically significant cycle. The top window of figure 2 plots observed cross country consumption as the solid (blue) line and smoothed cross-country TFP as the dot-dot (red) line. Not surprisingly, the volatility of cross-country consumption dominates cross-country TFP fluctuations. The standard deviation of the latter is 0.59 compared to 0.27 for the latter.

Nonetheless, the posterior mean of cross country TFP has an economically interesting story to tell. Cross-country TFP is flat in the latter 1970s, which reflects the productivity slowdown in the U.S. and catch up by Canada. By the 1980s, U.S. TFP is growing more rapidly than in Canada. This continues until the early 1990s, when Canadian TFP again recovers relative to U.S. TFP. At the end of the sample, the U.S.-Canadian TFP differential is expanding once more.

The plot of smoothed $\widetilde{c}_t$ appears in the bottom window of figure 2. The cycle in $\widetilde{c}_t$ is

[11] The Kalman smoother as described in Hamilton (1994).

apparent and shows the impact of movements in cross-country TFP. The posterior mean of $\tilde{c}_t$ is persistent and volatile. Its standard deviation is 2.01, while the leading term of the ACF gives a half-life to an own shock for $\tilde{c}_t$ of nearly ten quarters.

The cycle of $\tilde{c}_t$ has peaks and troughs that coincide with several U.S.-Canadian business cycles dates. For example, troughs in the posterior mean of $\tilde{c}_t$ appear in 1981 and 1990 which also represent recessions dates in the U.S. and Canada. Since the end of the 1990 – 1991 recession, the rise in $\tilde{c}_t$ points to persistent, but transitory, increase in U.S. consumption relative to Canada. However, $\tilde{c}_t$ has been falling rapidly since a peak in $2001Q3$, which corresponds to the end of the 2001 recession.

Figure 3 contain plots of the Canadian dollar-U.S. dollar exchange rate, its smoothed trend, and its smoothed cycle. The exchange rate is dominated by its trend. Trend volatility is almost 2.5 times greater than observed in the transitory component of the exchange rate. Note also that the exchange rate provides the smallest standard deviation for a transitory component as shown in table 3. The transitory component of the exchange rate is persistent because the leading term of its ACF is 0.92. This persistence is directly tied to $\tilde{c}_t$ because its correlation with the transitory component of the exchange rate equals -0.99. Exchange rate trend growth and cross-country TFP growth are also negatively correlated at -0.87. Replacing $\tilde{c}_t$ and cross-country TFP growth with $\widetilde{m}_t$ and m_t, yields correlations only of 0.22 and 0.31, respectively.

The strong negative correlation of the transitory component of the exchange rate with $\tilde{c}_t$ help to interpret the Canadian dollar-U.S. dollar exchange rate cycle. Peaks in the transitory component of the Canadian dollar-U.S. dollar exchange rate occur either at during or shortly after the end of recession. For example, the transitory component of the exchange rate peaks during the 1990 – 1991 recession, which is the last time the Canadian dollar approached par against the U.S. dollar. An exception is the end of the 2001 recession at which the Canadian reached a low of nearly 0.6 to the U.S. dollar. Thus, the transitory component of the exchange has economic content at the posterior mean of Γ which includes $\beta = 0.96$.

5c. Exchange Rate Dynamics as $\beta \longrightarrow 1$

Engel and West (2005) argue that the exchange rate will approximate a random walk when the discount factor is close to one and fundamentals have a unit root. The posterior distribution of β contains information about how close it needs to be to one to generate approximate random walk behavior in the exchange rate. For example, Proposition 9 shows that as $\beta \longrightarrow 1$ the transitory component of the exchange will collapse to zero at each date in the $1976Q1$ – $2004Q4$ sample.

Figure 4 plots the smoothed transitory component of the exchange rate given a specific draw from the posterior distribution of Γ. We sample from the posterior distribution conditioning on the smallest, 16th percentile, 84th percentile, and largest draws of β. These are $\beta = [0.906\ 0.944\ 0.978\ 0.999]$. The plots show that as β moves from 0.906 to 0.999 the volatility of is compressed to zero. For $\beta = 0.999$, the transitory component of the exchange rate is almost identically zero moving pointwise through the sample data. Plots of the transitory component of the exchange rate are economically interesting when draws from Γ produce a β below the posterior mean. Thus, it is most likely difficult for the data to distinguish between an independent random walk and the restrictions the DSGE-PVM model imposes.

6. Conclusion

Economists have little to say about the impact of policy on currency markets without a theory of exchange rate determination that is empirically relevant. We present theoretical results showing that the workhorse present-value model (PVM) of exchange rates places *com-*

mon trend and *common cycle* restrictions on the exchange rate and its fundamental. Assuming that the interest (semi-)elasticity of money demand is large and the common trend restriction, we show that the exchange rate approximates a martingale, consistent with recent findings by Engel and West (2005). Thus, the workhorse PVM of the exchange rate can explain the random walk behavior of actual exchange rates and helps us to understand why the naive random walk remains a compelling benchmark against which other exchange rate models are measured.

This paper also presents a PVM model of exchange rates based on a dynamic stochastic general equilibrium (DSGE) model. The DSGE model yields a PVM that places a richer set of predictions on the exchange rate and its fundamental. Under the DSGE-PVM, currency returns, money growth, and consumption growth are co-dependent in the sense of Vahid and Engle (1997). Along with a common trend restriction, the co-dependent restriction provides a collection of cross-equation restriction that appear testable. However, we show that as the discount factor of the DSGE-PVM approaches one the exchange is dominated by permanent shocks. This resembles the Engel and West (2005) result that the exchange is observationally equivalent to a random walk at large discount factors when fundamentals have a unit root.

Our empirical results support the contention that it is difficult for the data to choose between a exchange rate models when the discount factor is close to one. Preliminary estimates of the DSGE-Model suggest that data places similar weight on discount factors of 0.99 as on 0.96. At the latter estimate, the transitory component of the exchange has economic and statistical significance, while at the former it does not.

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Table 1: Estimates of the UC-Models

Posterior Means

Parameter	Priors	Two Cycles	Money Cycle	Consumption Cycle
β	Normal $[0.95,\ 0.025]$	0.96 (0.02)		
θ_1	Normal $[0.7,\ 0.2]$	0.91 (0.05)	–	
θ_2	Normal $[-1.0,\ 0.3]$	0.04 (0.05)	–	
α_1	Normal $[0.4,\ 0.2]$	0.54 (0.05)		–
α_2	Normal $[0.2,\ 0.1]$	0.33 (0.05)		–
μ^*	Normal $[-0.2,\ 0.1]$	-0.17 (0.07)		
a^*	Normal $[0.1,\ 0.1]$	0.18 (0.23×10^{-2})		
σ_μ	Uniform $[0.0,\ 2.0]$	1.53 (0.14)		
σ_A	Uniform $[0.0,\ 1.0]$	0.30 (0.03)		
σ_m	Uniform $[0.0,\ 2.0]$	1.67 (0.13×10^{-2})		
σ_c	Uniform $[0.0,\ 1.0]$	0.70 (0.01)		
$\rho_{A,c}$	Normal $[-0.5,\ 0.2]$	-0.60 (0.06)	–	
μ_e	Normal $[100.0,\ 15.0]$	125.28 (6.89)		
τ_e	Normal $[1.0,\ 0.5]$	1.65 (0.15)		
δ_A	Uniform $[-10.0,\ 0.0]$	-8.07 (0.31)		

For the parameters with a normal prior, the first value in brackets is the degenerate prior and the second the prior standard deviation. Priors for the θs are on the unconstrained AR coefficients. The associated posterior means are for constrained coefficients.

Table 2: Estimates of the UC-Models

Posterior Means

Parameter	Two Cycles	Money Cycle	Consumption Cycle
$\delta_{m,0}$	−0.93 (0.03)		
$\delta_{m,1}$	−0.50 (0.05)		
$\delta_{m,2}$	−0.32 (0.04)		
$\delta_{c,0}$	0.43 (0.16)		
$\delta_{c,1}$	0.02 (0.02)		

Table 3: Summary of the Posterior of the UC-Models

	Two Cycles	Money Cycle	Consumption Cycle
$STD(\Delta e^{trend})$	2.24		
$STD(e^{cycle})$	0.94		
$AR1(e^{cycle})$	0.92		
$Corr(\Delta e^{trend},\ e^{cycle})$	-0.17		
$STD(\Delta\mu)$	1.13		
$STD(\widetilde{m})$	1.35		
$AR1(\widetilde{m})$	-0.09		
$Corr(\Delta\mu,\ \widetilde{m})$	0.38		
$STD(\Delta\ln[A])$	0.27		
$STD(\widetilde{c})$	2.01		
$AR1(\widetilde{c})$	0.93		
$Corr(\Delta\ln[A],\ \widetilde{c})$	-0.27		
$Corr(\Delta e^{trend},\ \Delta\mu)$	0.31		
$Corr(\Delta e^{trend},\ \Delta\ln[A])$	-0.87		
$Corr(e^{cycle},\ \widetilde{m})$	0.22		
$Corr(e^{cycle},\ \widetilde{c})$	-0.99		

The summary statistics are taken from the mean of the posterior distributions of the trends and cycle of the exchange rate, cross-country money, and cross-country consumption.

Figure 1: Canadian-U.S. Money Differential Trend and Cycle

Figure 2: Canadian-U.S. Consumption Differential Trend and Cycle

Figure 3: CDN$/US$ Exchange Rate Trend and Cycle

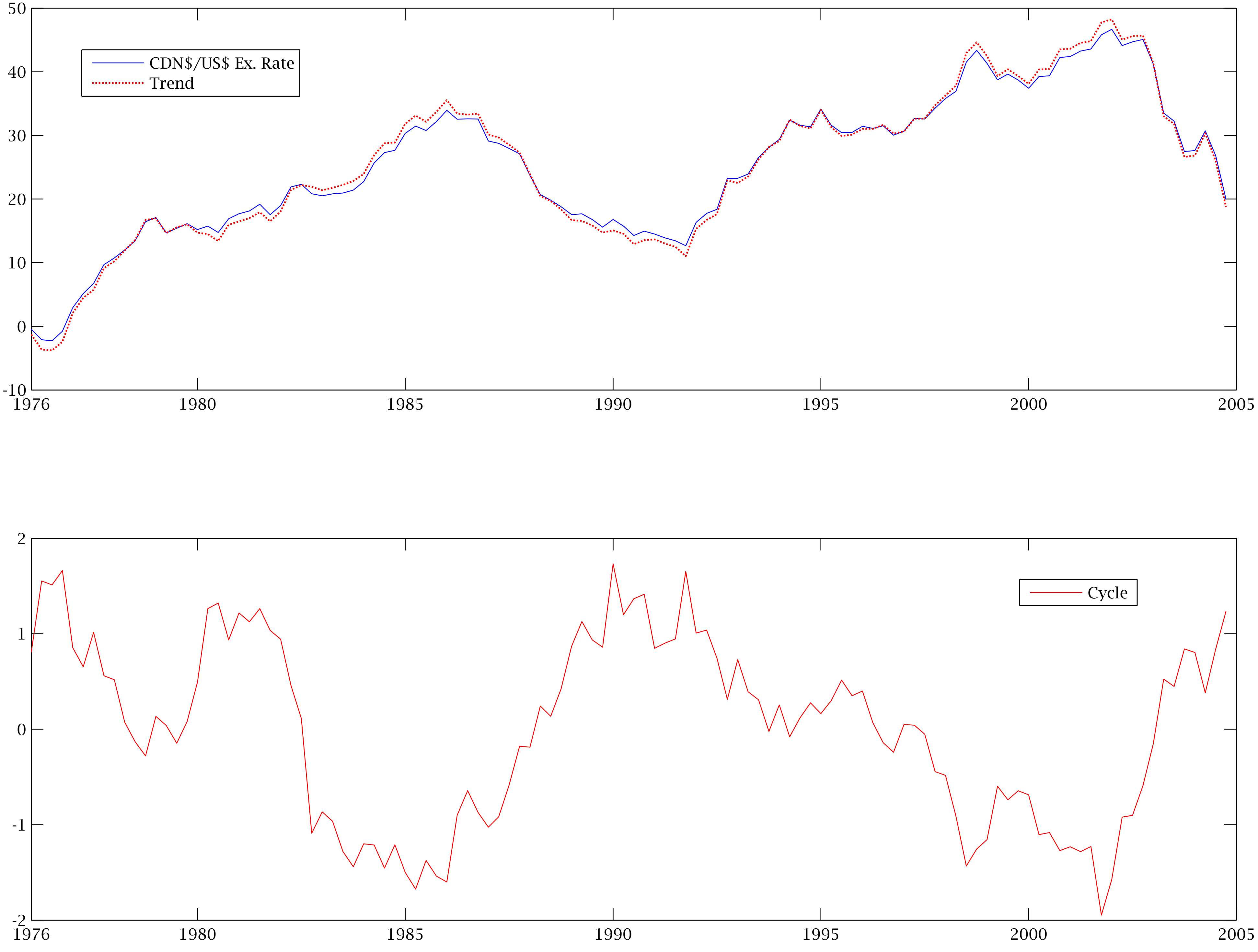

Figure 4: CDN$/US$ Exchange Rate Cycle at Discount Factor Estimates